Aging at criticality in model-*C* dynamics

P. Calabrese* and A. Gambassi†
Scuola Normale Superiore and INFN, Piazza dei Cavalieri 7, I-56126 Pisa, Italy
(Received 8 November 2002; published 20 March 2003)

We study the off-equilibrium two-point critical response and correlation functions for the relaxational dynamics with a coupling to a conserved density (model C) of the $O(N)$ vector model. They are determined in an $\epsilon=4-d$ expansion for vanishing momentum. We briefly discuss their scaling behaviors and the associated scaling forms are determined up to first order in ϵ. The corresponding fluctuation-dissipation ratio has a nontrivial large time limit in the aging regime and, up to one-loop order, it is the same as that of the model A for the physically relevant case $N=1$. The comparison with predictions of local scale invariance is also discussed.

DOI: 10.1103/PhysRevE.67.036111 PACS number(s): 64.60.Ht, 05.40.−a, 75.40.Gb, 05.70.Jk

I. INTRODUCTION

Nonequilibrium dynamics of statistical systems is currently under intensive theoretical investigation, and new dynamical behaviors have been recently discovered in models of disordered systems. One of the most striking of them is *aging*, i.e., a persistence of the system in a nonequilibrium state even after a macroscopic time has elapsed since the latest perturbation acting on it. As a consequence, there is no ''memory loss'' of the thermal history of the system and its response to an external field, for example, will depend on it. This fact is commonly observed in glassy systems [1,2]. It has been pointed out [3], however, that this kind of behavior may be also observed in critical nondisordered models. In these cases the presence of slow-relaxing modes could keep the system in a nonequilibrium state even asymptotically for large times. Consider, indeed, a system in a generic configuration and, at time $t=0$, bring it in contact with a thermal bath at a given temperature T. The resulting relaxation process is characterized by a transient behavior with off-equilibrium evolution, for $t<\tau_R$, and a stationary equilibrium evolution for $t>\tau_R$, where τ_R is the relaxation time. In the former the behavior of the system is expected to depend on initial conditions, while in the latter time homogeneity and time reversal symmetry (at least in the absence of external fields) are recovered and such a dependence is lost; fluctuations are thus described in terms of ''equilibrium'' dynamics.

In the following we focus on ferromagnetic systems quenched at their critical temperature T_c [4] for $t=0$ (interesting behaviors are observed also in the case of noninstantaneous quench, i.e., for time-dependent thermal bath [5]). A convenient way of describing dynamics is to study two-time response and correlation functions. The former is usually defined as $R_{\mathbf{x}}(t,s)=\delta\langle\varphi_{\mathbf{x}}(t)\rangle/\delta h(s)$, where φ is the magnetic order parameter, h is a small external field applied at time $s>0$ in the point $\mathbf{x}=\mathbf{0}$, and $\langle\cdot\rangle$ stands for the mean over the stochastic dynamics. The latter, instead, is defined as the order-parameter correlation function $C_{\mathbf{x}}(t,s)=\langle\varphi_{\mathbf{x}}(t)\varphi_{\mathbf{0}}(s)\rangle$.

If the system does not reach the equilibrium, the response and correlation functions will depend both on s (the ''age'' of the system, also called ''waiting time'') and on the observation time t. To characterize the distance from equilibrium of an aging system, evolving at a fixed temperature T, the fluctuation-dissipation ratio (FDR) is usually introduced [6,3],

$$X_{\mathbf{x}}(t,s)=\frac{T\,R_{\mathbf{x}}(t,s)}{\partial_s C_{\mathbf{x}}(t,s)}. \tag{1.1}$$

When t and the waiting time s are both greater than τ_R, the dynamics is homogeneous in time and time-reversal invariant so that the fluctuation-dissipation theorem can be applied, leading to $X_{\mathbf{x}}(t,s)=1$. This is no longer true in the aging regime [3]. It has been argued that the long-time limit of the FDR at criticality

$$X^\infty=\lim_{s\to\infty}\lim_{t\to\infty}X_{\mathbf{x}=0}(t,s) \tag{1.2}$$

is a novel *universal* quantity of nonequilibrium critical dynamics [7–9]. Correlation and response functions have been exactly computed for a random walk, a free Gaussian field, and a two-dimensional XY model at zero temperature and the value $X^\infty=1/2$ has been found [3]. In the case of the d-dimensional spherical model [8], one-dimensional Ising-Glauber chain [10,7] and two- and three-dimensional Ising model, investigated by Monte Carlo simulations [8], X^∞ has values ranging between 0 and $\frac{1}{2}$.

Field-theoretical methods have been proven a powerful tool for the computation of *universal* quantities (such as critical exponents) in critical phenomena (for an updated review see Ref. [11]). In this framework the problem of critical relaxation from a macroscopically prepared initial state has been analyzed since some years, and a new universal exponent associated with it has been introduced as a consequence of an additional time-surface renormalization [12].

We would take advantage of these previous works to compute the critical FDR and the associated *universal* scaling functions for mesoscopic models of dynamics, overcoming

*Email address: calabres@df.unipi.it
†Email address: andrea.gambassi@sns.it

most of the analytical difficulties encountered in the exact solutions of models with aging dynamics. In Refs. [13,14] this problem has been addressed for the dissipative dynamics (model A of Ref. [15]) of the $O(N)$ ferromagnetic model, whereas the purely dissipative dynamics of the diluted Ising model has been analyzed in Ref. [16]. Here we consider the $O(N)$ model dynamically coupled to a conserved density (model C of Ref. [15]). Physical realizations of this model are, e.g., intermetallic alloys [17], adsorbed layers on solid substrates [18] and supercooled liquid [19]. Also the deterministic microcanonical φ^4 model [20,21] is believed to be in the model C universality class since the order parameter is coupled to the conserved energy [22].

The paper is organized as follows. In Sec. II model C is introduced and its scaling forms are discussed. In Sec. III we derive the first order contribution in an ϵ expansion to the response and correlation functions for all values of s and t and we derive the FDR up to the same order. Finally in Sec. IV we discuss our results stressing their relevance for the issue (of applicability) of local scale invariance.

II. MODEL *C*

Let us consider the relaxational dynamics of an N-component field $\varphi(\mathbf{x},t)$ coupled to a noncritical conserved density $\varepsilon(\mathbf{x},t)$. This system may be described by means of the following coupled stochastic Langevin equations (model C of Ref. [15])

$$\partial_t\varphi(\mathbf{x},t) = -\Omega\frac{\delta\mathcal{H}[\varphi,\varepsilon]}{\delta\varphi(\mathbf{x},t)} + \xi(\mathbf{x},t), \tag{2.1}$$

$$\partial_t\varepsilon(\mathbf{x},t) = \Omega\rho\nabla^2\frac{\delta\mathcal{H}[\varphi,\varepsilon]}{\delta\varepsilon(\mathbf{x},t)} + \zeta(\mathbf{x},t), \tag{2.2}$$

where $\mathcal{H}[\varphi,\varepsilon]$ is the Landau-Ginzburg Hamiltonian for the fields φ and ε with a coupling term between them

$$\mathcal{H}[\varphi,\varepsilon] = \int d^dx\left[\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}r_0\varphi^2 + \frac{1}{4!}g_0\varphi^4 + \frac{1}{2}\varepsilon^2 + \frac{1}{2}\gamma_0\varepsilon\varphi^2\right], \tag{2.3}$$

where Ω and ρ are the kinetic coefficients, $r_0 \propto T - T_c$, g_0 and γ_0 the bare coupling constants, $\xi(\mathbf{x},t)$ and $\zeta(\mathbf{x},t)$ zero-mean stochastic Gaussian noises with

$$\langle\xi_i(\mathbf{x},t)\xi_j(\mathbf{x}',t')\rangle = 2\Omega\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')\,\delta_{ij}, \tag{2.4}$$

$$\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t')\rangle = -2\rho\,\Omega\,\nabla^2\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'). \tag{2.5}$$

The coupling between $\varepsilon(\mathbf{x},t)$ and $\varphi(\mathbf{x},t)$ does not change the static properties of the latter as it can be seen by computing the effective Hamiltonian for the φ field (see Ref. [23]). Moreover ε-field static correlation functions are related to φ^2-field correlation functions.

Dynamical correlation functions, generated by the Langevin equations (2.1) and (2.2) and averaged over the noises ξ and ζ, may be obtained by means of the field-theoretical action [24,23]

$$S[\varphi,\tilde{\varphi},\varepsilon,\tilde{\varepsilon}] = \int dt\int d^dx\left[\tilde{\varphi}\partial_t\varphi + \Omega\tilde{\varphi}\frac{\delta\mathcal{H}[\varphi,\varepsilon]}{\delta\varphi} - \tilde{\varphi}\Omega\tilde{\varphi} + \tilde{\varepsilon}\partial_t\varepsilon - \rho\,\Omega\tilde{\varepsilon}\nabla^2\frac{\delta\mathcal{H}[\varphi,\varepsilon]}{\delta\varepsilon} + \tilde{\varepsilon}\rho\,\Omega\,\nabla^2\tilde{\varepsilon}\right], \tag{2.6}$$

where $\tilde{\varphi}(\mathbf{x},t)$ and $\tilde{\varepsilon}(\mathbf{x},t)$ are the response fields associated with $\varphi(\mathbf{x},t)$ and $\varepsilon(\mathbf{x},t)$, respectively. It is easy to read from Eqs. (2.6) and (2.3) the interaction vertices, given by $-\Omega g_0\tilde{\varphi}\varphi^3/3!$, as in the case of model A, $-\Omega\gamma\varepsilon\tilde{\varphi}\varphi$ and $\rho\Omega\gamma\,\varphi^2\nabla^2\tilde{\varepsilon}/2$.

In Refs. [12,25] this formalism was extended to deal with relaxation of the system from a macroscopically prepared initial state. To take into account the effect of such initial condition on the dynamics described by Eq. (2.6), one has also to average over the possible initial configurations of both the order-parameter $\varphi_0(\mathbf{x}) = \varphi(\mathbf{x},t=0)$ and the conserved density $\varepsilon_0(\mathbf{x}) = \varepsilon(\mathbf{x},t=0)$ with a probability distribution $e^{-H_0[\varphi_0,\varepsilon_0]}$ given by [25]

$$H_0[\varphi_0,\varepsilon_o] = \int d^dx\left[\frac{\tau_0}{2}[\varphi_0(\mathbf{x}) - u(\mathbf{x})]^2 + \frac{1}{2c_0}[\varepsilon_0(\mathbf{x}) - v(\mathbf{x})]^2\right]. \tag{2.7}$$

This specifies an initial state $u(\mathbf{x})$ for $\varphi(\mathbf{x},t)$ and $v(\mathbf{x})$ for $\varepsilon(\mathbf{x},t)$ with correlations proportional to τ_0^{-1} and c_0, respectively. Response and correlation functions may be obtained, following standard methods [24,23], by a perturbative expansion of the functional weight $e^{-(S[\varphi,\tilde{\varphi},\varepsilon,\tilde{\varepsilon}]+H_0[\varphi_0,\varepsilon_0])}$. An initial condition with long-range correlations may lead to a different universality class, as, e.g., shown for the d-dimensional spherical model with nonconservative dynamics [26].

The propagators (Gaussian two-point correlation and response functions) of the resulting theory are [25]

$$\langle\tilde{\varphi}_i(\mathbf{q},s)\varphi_j(-\mathbf{q},t)\rangle_0 = \delta_{ij}R^0_q(t,s) = \delta_{ij}\;\theta(t-s)G(t-s), \tag{2.8}$$

$$\langle\varphi_i(\mathbf{q},s)\varphi_j(-\mathbf{q},t)\rangle_0 = \delta_{ij}C^0_q(t,s) = \frac{\delta_{ij}}{q^2+r_0}\left[G(|t-s|) + \left(\frac{r_0+q^2}{\tau_0} - 1\right)G(t+s)\right], \tag{2.9}$$

where

$$G(t) = e^{-\Omega(q^2+r_0)t}, \tag{2.10}$$

and [25]

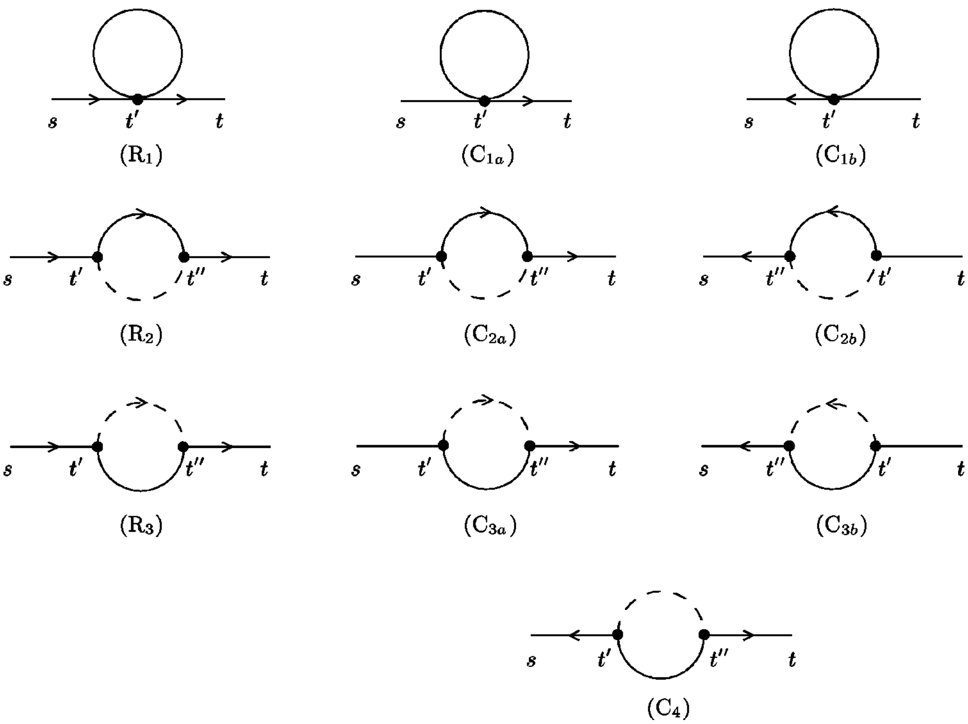

FIG. 1. Feynman diagrams contributing to the one-loop order-parameter response [(R_1), (R_2), (R_3)] and correlation function [$(\mathrm{C}_{1a,b})$, $(\mathrm{C}_{2a,b})$, $(\mathrm{C}_{3a,b})$, (C_4)]. Response functions are drawn as lines with arrows going from the early time to the later one, whereas correlators bear no arrow lines. Solid (dashed) lines refer to the order-parameter φ (to the conserved density ε).

$$\langle \tilde{\varepsilon}(\mathbf{q},s)\varepsilon(-\mathbf{q},t)\rangle_0 = R^0_{\varepsilon,q}(t,s) = \theta(t-s)G_\varepsilon(t-s), \tag{2.11}$$

$$\langle \varepsilon(\mathbf{q},s)\varepsilon(-\mathbf{q},t)\rangle_0 = C^0_{\varepsilon,q}(t,s) = G_\varepsilon(|t-s|) + (c_0-1)G_\varepsilon(t+s), \tag{2.12}$$

with

$$G_\varepsilon(t) = e^{-\rho\Omega(q^2+r_0)t}. \tag{2.13}$$

As in the case of model *A* and model *B*, it has been shown that τ_0^{-1} is irrelevant (in the renormalization group sense) so that we set $\tau_0^{-1}=0$ [12,25].

A. Scaling forms

When a ferromagnetic system is quenched from a disordered initial state to its critical point, the correlation length grows as $t^{1/z}$, where z is the dynamical critical exponents [15] and t the time elapsed since the quench. So in momentum space, applying standard scaling arguments, the universal two-time (s,t) response and correlation functions depend only on the two products $q^z t$ and $q^z s$, where q is the external momentum.

In particular general renormalization group argument suggest the scaling forms [12,25]

$$\Omega R_{\mathbf{q}=\mathbf{0}}(t,s) = A_R(t-s)^a (t/s)^\theta F_R(s/t), \tag{2.14}$$

$$C_{\mathbf{q}=\mathbf{0}}(t,s) = A_C s(t-s)^a (t/s)^\theta F_C(s/t), \tag{2.15}$$

where $R_{\mathbf{q}}(t,s)$ and $C_{\mathbf{q}}(t,s)$ are the Fourier transforms (with respect to $\mathbf{x}$) of $R_{\mathbf{x}}(t,s)$ and $C_{\mathbf{x}}(t,s)$, respectively, $a=(2-\eta-z)/z$ [27], and θ is the initial-slip exponent of response function [12,25]. The functions $F_C(v)$ and $F_R(v)$ are universal provided one fixes the nonuniversal normalization constant A_R and A_C to have $F_i(0)=1$.

In Ref. [13] the following quantity, related to the FDR, was introduced in momentum space

$$\mathcal{X}_{\mathbf{q}}(t,s) = \frac{\Omega R_{\mathbf{q}}(t,s)}{\partial_s C_{\mathbf{q}}(t,s)}. \tag{2.16}$$

It has been argued that the zero-momentum limit

$$\mathcal{X}^\infty_{\mathbf{q}=\mathbf{0}} = \lim_{s\to\infty}\lim_{t\to\infty} \mathcal{X}_{\mathbf{q}=\mathbf{0}}(t,s), \tag{2.17}$$

is equal to the same limit of the FDR (1.2) for $\mathbf{x}=\mathbf{0}$, i.e., $\mathcal{X}^\infty_{\mathbf{q}=\mathbf{0}} = X^\infty$ to all orders [13]. This fact allows an easier perturbative computation (in momentum space) of the new universal quantity X^∞. Combining scaling forms and previous definitions, we find

$$\mathcal{X}^\infty_{\mathbf{q}=0}(t,s) = \lim_{s\to\infty}\lim_{t\to\infty} \frac{T_c R_{\mathbf{x}=0}(t,s)}{\partial_s C_{\mathbf{x}=0}(t,s)} = \frac{A_R}{A_C(1-\theta)}. \tag{2.18}$$

In recent works the notion of local scale invariance has been introduced as an extension of anisotropic or dynamical scaling (see Ref. [28] and references therein). Assuming the covariance of the response function under a suitable subgroup of the constructed group of local scale transformations, it has been argued that [28]

$$R_{\mathbf{x}}(t,s) = R_{\mathbf{x}=\mathbf{0}}(t,s)\Phi(|\mathbf{x}|/(t-s)^{1/z}), \tag{2.19}$$

where [29]

$$R_{\mathbf{x}=\mathbf{0}}(t,s)=\mathcal{A}_R(t-s)^{a'}(t/s)^{\theta}, \tag{2.20}$$

and $\Phi(u)$ is a function whose explicit and convergent series expansion is known [28]. Fourier transforming Eq. (2.19) and setting $\mathbf{q}=0$ one could obtain the strong prediction $F_R(s/t)=1$. For the correlation function and its derivative no analogous result exists.

III. ONE-LOOP FDR

In this section we compute the nonequilibrium critical two-point response and correlation functions for the model C up to one-loop order, for vanishing external momentum. We use here the method of renormalized field theory in the minimal subtraction scheme. The breaking of time homogeneity gives rise to some technical problems in the renormalization procedure in terms of one-particle irreducible correlation functions [12] so our computation is done in terms of connected functions.

At one-loop order we have to evaluate, taking also into account causality [24], the ten Feynman diagrams depicted in Fig. 1, three for the response function [(R_1), (R_2), and (R_3)] and seven for the correlation one [($\mathrm{C}_{1a,b}$), ($\mathrm{C}_{2a,b}$), ($\mathrm{C}_{3a,b}$), and (C_4)].

In terms of them we have

$$R_q(t,s)=R_q^0(t,s)-\frac{N+2}{6}g_0(\mathrm{R}_1)+\Omega^2\gamma^2(\mathrm{R}_2)+\rho\Omega^2\gamma^2(\mathrm{R}_3)+O(g_0^2,g_0\gamma^2,\gamma^4), \tag{3.1}$$

$$C_q(t,s)=C_q^0(t,s)-\frac{N+2}{6}g_0[(\mathrm{C}_{1a})+(\mathrm{C}_{1b})]+\Omega^2\gamma^2[(\mathrm{C}_{2a})+(\mathrm{C}_{2b})]+\rho\Omega^2\gamma^2[(\mathrm{C}_{3a})+(\mathrm{C}_{3b})]+\rho\Omega^2\gamma^2(\mathrm{C}_4)+O(g_0^2,g_0\gamma^2,\gamma^4). \tag{3.2}$$

In order to evaluate the FDR at criticality we set, in this perturbative expansion, $r_0=0$ (massless theory). We also set $\tau_0^{-1}=0$, since it is an irrelevant variable [12], and $\Omega=1$ to lighten the notations. The first step in the calculation of the diagrams is the evaluation of the critical "bubbles" $B_{1c}(t)$, $B_{2c}(t',t'')$, $B_{3c}(t',t'')$, and $B_{4c}(t',t'')$, i.e., the one-particle irreducible parts common to diagrams depicted on the first, second, third, and fourth line of Fig. 1, respectively. We have, in generic dimension d [13]

$$B_{1c}(t)=\int\frac{d^dq}{(2\pi)^d}C_q^0(t,t)=-\frac{1}{d/2-1}\frac{(2t)^{1-d/2}}{(4\pi)^{d/2}}=-N_d\frac{\Gamma(d/2-1)}{2^{d/2}}t^{1-d/2}, \tag{3.3}$$

where $N_d=2/(4\pi)^{d/2}\Gamma(d/2)$. Given we are interested in the value of the FDR (2.16) for $\mathbf{q}=\mathbf{0}$ we evaluate, in the following, all diagrams for vanishing external momentum. Then for $B_{2c}(t,s)$, $B_{3c}(t,s)$, and $B_{4c}(t,s)$ we have

$$B_{2c}(t,s)=\int\frac{\mathrm{d}^dq}{(2\pi)^d}R_q^0(t,s)C_{\varepsilon,q}^0(t,s)=\theta(t-s)[4\pi\Omega(1+\rho)]^{-d/2}[(t-s)^{-d/2}+(c_0-1)(t-\kappa s)^{-d/2}], \tag{3.4}$$

$$B_{3c}(t,s)=\int\frac{\mathrm{d}^dq}{(2\pi)^d}q^2R_{\varepsilon,q}^0(t,s)C_q^0(t,s)=\theta(t-s)[4\pi\Omega(1+\rho)]^{-d/2}[(t-s)^{-d/2}-(t+\kappa s)^{-d/2}], \tag{3.5}$$

$$B_{4c}(t>s,s)=\int\frac{\mathrm{d}^dq}{(2\pi)^d}C_{\varepsilon,q}^0(t,s)C_q^0(t,s)=\frac{N_d}{2}\Gamma(d/2-1)[\Omega(1+\rho)]^{1-d/2}\{(t-s)^{1-d/2}-(t+\kappa s)^{1-d/2}+(c_0-1)[(t-\kappa s)^{1-d/2}-(t+s)^{1-d/2}]\}, \tag{3.6}$$

where $\kappa=(1-\rho)/(1+\rho)<1$ (given that, for model C to make sense, $\rho>0$). Expression (3.6) for $B_{4c}(t,s)$ is valid only for $t>s$, that for $s>t$ is easily found, given the symmetry property $B_{4c}(t,s)=B_{4c}(s,t)$. Once critical bubbles have been determined, it is easy to compute each diagram in Fig. 1.

Performing the required integrations and expanding in powers of ϵ we find, for the bare response function,

$$R_{q=0}(t,s)=-\frac{2\tilde{\gamma}_0^2}{1+\rho}\frac{1}{\epsilon}+1+\left[\tilde{g}_0\frac{N+2}{24}-\tilde{\gamma}_0^2\frac{1+\rho^2-c_0}{2\rho(1+\rho)}\right]\times\ln\frac{t}{s}-\frac{\tilde{\gamma}_0^2}{1+\rho}\ln[\Omega(t-s)]-\tilde{\gamma}_0^2c_0\frac{1}{1-\rho^2}\ln\frac{1-\kappa v}{1-\kappa}+\tilde{\gamma}_0^2\mathcal{R}(s/t;\rho)+O(\epsilon^2,\tilde{g}_0^2,\epsilon\tilde{g}_0,\tilde{\gamma}^4,\tilde{\gamma}^2\tilde{g}_0,\epsilon\tilde{\gamma}^2), \tag{3.7}$$

where

$$\mathcal{R}(v;\rho)=-\frac{\rho}{1-\rho^2}\ln\frac{1+\kappa v}{2}+\frac{1}{1-\rho^2}\ln\frac{1-\kappa v}{2\rho}-\frac{1}{1+\rho}, \tag{3.8}$$

and for the correlation function

$$C_{q=0}(t,s) = -\frac{4\tilde{\gamma}_0^2 \Omega s}{1+\rho}\frac{1}{\epsilon} + 2\Omega s\Bigg\{1+\tilde{g}_0\frac{N+2}{12} + \left[\tilde{g}_0\frac{N+2}{24} - \tilde{\gamma}_0^2\frac{1+\rho^2-c_0}{2\rho(1+\rho)}\right] \times \ln\frac{t}{s} - \frac{\tilde{\gamma}_0^2}{1+\rho}\ln[\Omega t] + \tilde{\gamma}_0^2\Bigg[-c_0\mathcal{C}_1\left(\frac{s}{t};\rho\right) - c_0\mathcal{C}_2(\rho) - \frac{\ln[1+\rho]}{1+\rho} + \mathcal{C}_2(\rho) + \frac{1}{\rho}\mathcal{C}_2\left(\frac{1}{\rho}\right) + \mathcal{C}_1\left(\frac{s}{t};\rho\right) - \mathcal{C}_1\left(-\frac{s}{t};\rho\right)\Bigg]\Bigg\} + O(\epsilon^2,\tilde{g}_0^2,\epsilon\tilde{g}_0,\tilde{\gamma}^4,\tilde{\gamma}^2\tilde{g}_0,\epsilon\tilde{\gamma}^2), \tag{3.9}$$

where we assumed $t>s$ and we introduced $\tilde{g}_0 = N_d g_0$, $\tilde{\gamma}_0 = N_d\gamma_0$ and the functions

$$\mathcal{C}_1(v;\rho) = \frac{1+v}{2v(1+\rho)}\ln[1+v] - \frac{1-\kappa v}{2v(1+\rho)\kappa^2}\ln[1-\kappa v], \tag{3.10}$$

$$\mathcal{C}_2(\rho) = -\frac{\ln[1-\kappa]}{(1-\rho)^2} - \frac{1}{(1-\rho)\rho}. \tag{3.11}$$

The first one is defined for $-1<v<1/\kappa$ and $\rho\neq 1$ (we are interested only in the case $\rho\geqslant 0$). We note that contributions to Eq. (3.9) coming from $\mathcal{C}_i$ are regular in the limit $\rho\to 1$.

The previous expressions for $R_{q=0}$ and $C_{q=0}$ have simple poles in ϵ, so renormalization of the bare parameters is required. We use the minimal subtraction scheme in order to render renormalized quantities finite for $\epsilon\to 0$. At one-loop order it is sufficient to perform the following renormalizations [30,23]:

$$\tilde{\varphi}\mapsto\tilde{Z}^{-1/2}\tilde{\varphi},$$

$$\Omega\mapsto\tilde{Z}^{-1/2}\Omega \quad \text{with} \quad \tilde{Z} = 1-\frac{4\tilde{\gamma}^2}{1+\rho}\frac{1}{\epsilon} + O(\tilde{\gamma}^4,\tilde{\gamma}^2\tilde{g},\tilde{g}^2), \tag{3.12}$$

to render two-point functions finite, since $Z = 1+O(\tilde{g}^2)$ as known from statics [23].

Let us briefly recall the scenario of fixed points for out-of-equilibrium model C [30,23,25]. The fixed-point values for the couplings g and γ are determined only by the statics. We have $\tilde{g}^* = \tilde{g}_A^* + 6\tilde{\gamma}^{2*}$, where $\tilde{g}_A^* = 6\epsilon/(N+8) + O(\epsilon^2)$ is the fixed-point value of the coupling constant for model A [23].

The value of γ at the infrared stable fixed point depends on the sign of the specific-heat exponent α,

$$\tilde{\gamma}^{2*} = \begin{cases} 0 & \text{stable for } \alpha<0, \quad \text{case(I)}, \\ \dfrac{4-N}{N(N+8)}\epsilon + O(\epsilon^2) & \text{stable for } \alpha>0, \quad \text{case(II)}, \end{cases} \tag{3.13}$$

in the case (I), the dynamics of the conserved density decouples from that of the order parameter and we get back to model A (at least asymptotically). At the leading order in ϵ expansion we have, for the $O(N)$ model [11],

$$\alpha = \frac{4-N}{2(N+8)}\epsilon + O(\epsilon^2), \tag{3.14}$$

thus the truly model C dynamical fixed point is stable for $N<4+O(\epsilon)$. In three dimensions, numerical calculations shows [11] that α is negative already for $N=2$, so that the model C dynamics may be realized only for the three-dimensional Ising model ($N=1$) that has positive α [11] (the two-dimensional Ising model has $\alpha=0$ and the values of dynamical critical exponents for model C are still debated [31]).

As far as ρ is concerned we have three possible stable fixed points determined by equilibrium dynamics [30] (a) $\rho^*=\infty$, stable for $N>N_1(\epsilon) = 4-[15/4+3/2\ln(4/3)]\epsilon + O(\epsilon^2)$; (b) $\rho^* = 2/N-1+O(\epsilon)$, stable for $N<2+C\epsilon|\ln(\epsilon)|$ and for $N_2<N<N_1$, where $N_2(\epsilon) = 4-[7/2+3\ln(4/3)]\epsilon+O(\epsilon^2)$; and (c) $\rho^*=0$, which governs the critical behavior in the complement of the two regions, but it is a peculiar limit [23].

Finally, regarding the out-of-equilibrium dynamics, it has been shown that, whenever $\alpha>0$, the fixed point value for c is $c^*=0$ [25].

We focus our attention on the only relevant stable fixed point of the model, i.e., (IIb), for which

$$\tilde{g}^* = \frac{24}{N(N+8)}\epsilon + O(\epsilon^2). \tag{3.15}$$

Taking into account scaling forms (2.14) and (2.15), we find the well-known critical exponents for model C [23,25] (some of these results have been corrected at two-loop order in Ref. [32])

$$\theta = \tilde{g}^*\frac{N+2}{24} - \tilde{\gamma}^{2*}\frac{1+\rho^{*2}}{2\rho^*(1+\rho^*)} + O(\epsilon^2) = \frac{N^2-8N+10}{2(N-2)(N+8)}\epsilon + O(\epsilon^2),$$

$$\frac{2-\eta-z}{z} = -\frac{\tilde{\gamma}^{2*}}{1+\rho^*} + O(\epsilon^2) = -\frac{4-N}{2(N+8)}\epsilon + O(\epsilon^2), \tag{3.16}$$

and the scaling functions F_R and F_C are easily identified in Eqs. (3.7) and (3.9) with $c^*=0$,

$$F_R(v) = 1+\tilde{\gamma}^{*2}[\mathcal{R}(v;\rho^*) - \mathcal{R}(0;\rho^*)] + O(\epsilon^2), \tag{3.17}$$

$$F_C(v)=1+\tilde{\gamma}^{2*}\Bigg[\frac{1}{1+\rho^*}\ln(1-v)+\mathcal{C}_1(v;\rho^*)-\mathcal{C}_1(-v;\rho^*)\Bigg]+O(\epsilon^2). \tag{3.18}$$

In particular substituting fixed point values, we obtain the scaling form (we remember that $0\leqslant v\leqslant 1$ and $N<2+O(\epsilon)$ so that no worries about the sign in the argument of the second logarithm arise)

$$F_R(v)=1+\frac{4-N}{4(N+8)(N-1)}\epsilon[(N-2)\ln(1+(N-1)v)+N\ln(1-(N-1)v)]+O(\epsilon^2), \tag{3.19}$$

and, for the physically relevant case of $N=1$,

$$F_R(v)=1-\epsilon\frac{v}{6}+O(\epsilon^2), \tag{3.20}$$

that displays a correction to the mean-field value already at one-loop order (at variance with model *A* [13]).

We are now in the position to evaluate the FDR for model *C*. We first note that its Gaussian expression is the same as that of model *A* as far as φ and $\tilde{\varphi}$ are concerned and of model *B* (with some straightforward changes due to noncritical behavior of the conserved field) for ε and $\tilde{\varepsilon}$. In order to evaluate the φ-field FDR we compute the derivative with respect to *s* of the two-time correlation function and consider its ratio with the response one:

$$\frac{1}{2}\mathcal{X}^{-1}_{q=0}(t,s)=1+\tilde{g}^*\frac{N+2}{24}+\tilde{\gamma}^{2*}\Bigg\{\frac{1}{1+\rho^*}\ln\frac{1-s/t}{1+\rho^*}+\mathcal{C}_2(\rho^*)+\frac{1}{\rho^*}\mathcal{C}_2\left(\frac{1}{\rho^*}\right)+\frac{1+\rho^{*2}}{2\rho^*(1+\rho^*)}+\mathcal{C}_1\left(\frac{s}{t};\rho^*\right)-\mathcal{C}_1\left(-\frac{s}{t};\rho^*\right)-\mathcal{R}\left(\frac{s}{t},\rho^*\right)+\frac{s}{t}\Bigg[\partial_1\mathcal{C}_1\left(\frac{s}{t},\rho^*\right)+\partial_1\mathcal{C}_1\left(-\frac{s}{t},\rho^*\right)\Bigg]\Bigg\}+O(\epsilon^2), \tag{3.21}$$

where ∂_1 stands for the derivative with respect to the first argument. Note that, at variance with the one-loop FDR of model *A*, this result depends on s/t.

In the limit $t\rightarrow\infty$, s fixed, we find an s-independent expression,

$$\frac{1}{2}(\mathcal{X}^{\infty}_{q=0})^{-1}=1+\tilde{g}^*\frac{N+2}{24}+\tilde{\gamma}^{2*}\Bigg[\frac{2\rho^*}{(1+\rho^*)(1-\rho^*)^2}\times\ln\frac{(1+\rho^*)^2}{4\rho^*}-\frac{1+\rho^*}{2\rho^*}\Bigg]+O(\epsilon^2). \tag{3.22}$$

Taking into account the fixed point values of couplings we find

$$\mathcal{X}^{\infty}_{q=0}=\frac{1}{2}\Bigg\{1+\frac{4-N}{N(N+8)}\epsilon\Bigg[\frac{N(N-1)}{(4-N)(2-N)}+\frac{N^2(2-N)}{4(N-1)^2}\ln[N(2-N)]\Bigg]\Bigg\}+O(\epsilon^2). \tag{3.23}$$

For $N=1$, which is the physically relevant case into which model *C* is nontrivial, the result is exactly the same as in model *A*, $\mathcal{X}^{\infty}_{q=0}=1/2(1-\epsilon/12)+O(\epsilon^2)$, i.e., the presence of a coupled conserved density does not affect the value of $\mathcal{X}^{\infty}_{q=0}$, at least up to one-loop order.

In the ϵ expansion for $N>N_1(\epsilon)=4-[15/4+3/2\ln(4/3)]\epsilon+O(\epsilon^2)$ a fixed point with $\rho^*=\infty$ governs the critical behavior of the systems, but it probably disappears in three dimensions. In this limit we find (considering always $\alpha>0$ to ensure $c_0=0$)

$$\frac{1}{2}(\mathcal{X}^{\infty}_{q=0})^{-1}=1+\tilde{g}^*_A\frac{N+2}{24}+\frac{N}{4}\tilde{\gamma}^{2*}+O(2\text{-loop}). \tag{3.24}$$

Once again, as it happens in all the models that have been considered so far in the literature, the loop corrections lead to an FDR that is less than the mean-field value 1/2.

IV. CONCLUSIONS

In this work we considered the off-equilibrium properties of the *N*-vector model coupled to a conserved energy density (model *C*) in the framework of the field theoretical ϵ expansion. We computed up to the first order in ϵ the critical FDR as a function of the waiting time *s* and of the observation time *t*. In the long-time limit, for the physically relevant case of one component (Ising model) this ratio has the same value as in purely dissipative model *A*. Higher-loop calculations may clarify whether this is only a coincidence at one-loop or it is a deeper property.

We also obtain the $O(\epsilon)$ expression for the response and correlation function for vanishing external momentum. In both the cases we found corrections to the mean-field forms. Thus the result for the response function apparently disagrees with the prediction of local scale invariance (see Ref. [28] for an exhaustive introduction), i.e., $F_R(v)=1$. The same disagreement was already noted at two-loop order in the response function of model *A* [14]. In that case, however, the presence of a very small prefactor in the correction makes very hard the detection of this effect both in experiments and in Monte Carlo simulations. In the present case, instead, for $N=1$, the correction in

$$F_R(v)=1-\epsilon\frac{v}{6}+O(\epsilon^2) \tag{4.1}$$

should be large enough to be detectable. A Monte Carlo simulation of model *C* could be helpful to clarify the nature of this disagreement. In Ref. [14] we stressed that some problems in the comparison between Eq. (4.1) and the predictions of local scale invariance Eq. (2.19) could be connected with the Fourier transformability of $R_{\mathbf{x}}(t,s)$, concluding that some insight could be obtained by looking at the full q dependence of $R_q(t,s)$. This dependence was too cumbersome to be carried out in the two-loop computation of model *A* and it is still a difficult task for the model-*C* dynamics at one loop. In a forthcoming work we analyze the full q dependence of $R_q(t,s)$ for a φ^3 theory, showing that no problem arises with the Fourier transform. The nature of such a disagreement should be probably found in the limits of applicability of local scale invariance.

ACKNOWLEDGMENT

We would thank Malte Henkel for useful comments, suggestions, and correspondence.

[1] L.C.E. Struik, *Physical Aging in Amorphous Polymers and Other Materials* (Elsevier, Amsterdam, 1978).

[2] E. Vincent, J. Hammann, M. Ocio, J.P. Bouchaud, and L.F. Cugliandolo, Lect. Notes Phys. **492**, 184 (1997); J.P. Bouchaud, L.F. Cugliandolo, J. Kurchan, and M. Mézard, in *Spin Glasses and Random Fields*, edited by A.P. Young Directions in Condensed Matter Physics, Vol. 12 (World Scientific, Singapore, 1998); L.F. Cugliandolo, e-print cond-mat/0210312.

[3] L.F. Cugliandolo, J. Kurchan, and G. Parisi, J. Phys. I **4**, 1641 (1994).

[4] We are not interested here in the problem of phase ordering dynamics (see Ref. [33] for a review), so we will not consider the case of a quench to $T<T_c$, that instead is considered in Ref. [34].

[5] A. Picone, M. Henkel, and J. Richert, J. Phys. A **36**, 1249 (2003).

[6] L.F. Cugliandolo and J. Kurchan, Phys. Rev. Lett. **71**, 173 (1993); J. Phys. A **27**, 5749 (1994).

[7] C. Godrèche and J.M. Luck, J. Phys. A **33**, 1151 (2000).

[8] C. Godrèche and J.M. Luck, J. Phys. A **33**, 9141 (2000).

[9] C. Godrèche and J.M. Luck, J. Phys.: Condens. Matter **14**, 1589 (2002).

[10] E. Lippiello and M. Zannetti, Phys. Rev. E **61**, 3369 (2000).

[11] A. Pelissetto and E. Vicari, Phys. Rep. **368**, 549 (2002) and references therein.

[12] H.K. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B: Condens. Matter **73**, 539 (1989); H.K. Janssen, in *From Phase Transitions to Chaos–Topics in Modern Statistical Physics*, edited by G. Györgyi, I. Condor, L. Saswári, T. Tel (World Scientific, Singapore, 1992).

[13] P. Calabrese and A. Gambassi, Phys. Rev. E **65**, 066120 (2002); Acta Phys. Slov. **52**, 311 (2002).

[14] P. Calabrese and A. Gambassi, Phys Rev. E **66**, 066101 (2002).

[15] P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[16] P. Calabrese and A. Gambassi, Phys. Rev. B **66**, 212407 (2002).

[17] V.I. Gorentsveig, P. Fratzl, and J.L. Lebowitz, Phys. Rev. B **55**, 2912 (1997).

[18] K. Binder, W. Kinzel, and D.P. Landau, Surf. Sci. **117**, 232 (1982).

[19] H. Tanaka, J. Phys.: Condens. Matter **11**, L159 (1999).

[20] L. Caiani, L. Casetti, C. Clementi, G. Pettini, M. Pettini, and R. Gatto, Phys. Rev. E **57**, 3886 (1998); L. Caiani, L. Casetti, and M. Pettini, J. Phys. A **31**, 3357 (1998).

[21] B. Zheng, M. Schulz, and S. Trimper, Phys. Rev. Lett. **82**, 1891 (1999).

[22] J. Kockelkoren and H. Chaté, Physica D **168-169**, 80 (2002).

[23] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed. (Clarendon Press, Oxford, 1996).

[24] R. Bausch, H.K. Janssen, and H. Wagner, Z. Phys. B: Condens. Matter **24**, 113 (1976).

[25] K. Oerding and H.K. Janssen, J. Phys. A **26**, 3369 (1993).

[26] A. Picone and M. Henkel, J. Phys. A **35**, 5575 (2002).

[27] We remark that the exponent a introduced here and in Refs. [13,14,16] is not the same as that used in Refs. [7,9,29].

[28] M. Henkel, Nucl. Phys. B **641**, 405 (2002).

[29] M. Henkel, M. Pleimling, C. Godrèche, and J.M. Luck, Phys. Rev. Lett. **87**, 265701 (2001).

[30] E. Brézin and C. De Dominicis, Phys. Rev. B **12**, 4954 (1975).

[31] D. Stauffer, Int. J. Mod. Phys. C **8**, 1263 (1997); P. Sen, S. Dasgupta, and D. Stauffer, Eur. Phys. J. B **1**, 107 (1998); G. Besold, W. Schleier, and K. Heinz, Phys. Rev. E **48**, 4102 (1993).

[32] R. Folk and G. Moser, Phys. Rev. Lett. **89**, 125301 (2002); Acta Phys. Slov. **52**, 285 (2002).

[33] A.J. Bray, Adv. Phys. **43**, 357 (1994).

[34] M. San Miguel, J.D. Gunton, G. Dee, and P. Sahni, Phys. Rev. B **23**, 2334 (1981); T. Ohta, K. Kawasaki, A. Sato, and Y. Enomota, Phys. Lett. A **126**, 93 (1987); A. Chakrabarti, J.B. Collins, and J.D. Gunton, Phys. Rev. B **38**, 6894 (1988); K.R. Elder, B. Morin, M. Grant, and R.C. Desai, *ibid.* **44**, 6673 (1991).